

On the asymptotic distribution of certain bivariate reinsurance treaties

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Abstract: Let $\{(X_n, Y_n), n \geq 1\}$ be bivariate random claim sizes with common distribution function F and let $\{N(t), t \geq 0\}$ be a stochastic process which counts the number of claims that occur in the time interval $[0, t], t \geq 0$. In this paper we derive the joint asymptotic distribution of randomly indexed order statistics of the random sample $(X_1, Y_1), (X_2, Y_2), \dots, (X_{N(t)}, Y_{N(t)})$ which is then used to obtain asymptotic representations for the joint distribution of two generalised largest claims reinsurance treaties available under specific insurance settings. As a by-product we obtain a stochastic representation of a m -dimensional Λ -extremal variate in terms of iid unit exponential random variables.

Key Words: Generalised largest claims reinsurance treaty, bivariate random order statistics, bivariate treaties, asymptotic results, extreme value theory, m -dimensional Λ -extremal variate.

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1 Introduction

Let $\{(X_n, Y_n), n \geq 1\}$ be bivariate random claim sizes with common distribution function F arising from an insurance portfolio. In a specific insurance context, X_n may stay for instance for the total claim amount related to the n th accident, and Y_n for the corresponding total expense amount. Let further $\{N(t), t \geq 0\}$ be a stochastic process which counts the number of claims that occur in the time interval $[0, t], t \geq 0$. So we observe $(X_1, Y_1), \dots, (X_{N(t)}, Y_{N(t)})$ claims up to time $t > 0$. Denote by $X_{i:N(t)}, Y_{i:N(t)}, 1 \leq i \leq N(t)$ the

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corresponding i th lowest claim in the above random sample taken component-wise.

Especially in reinsurance applications, the largest claims $X_{N(t)-i+1:N(t)}, Y_{N(t)-i+1:N(t)}, i = 1, \dots, m$ are of particular importance. For instance, consider the following reinsurance contract introduced by Ammeter (1964)

$$S_1(p, t) = X_{N(t):N(t)} + X_{N(t)-1:N(t)} + \dots + X_{N(t)-p+1:N(t)}, \quad p \geq 1$$

which is known as the Largest Claim Reinsurance. Thus at the time point t a reinsurer covers the total loss amount $S_1(p, t)$ which is the simplest linear transformation of the upper p largest claims observed up to time t . Another simple reinsurance treaty

$$S_1(p, t) = X_{N(t):N(t)} + X_{N(t)-1:N(t)} + \dots + X_{N(t)-p+2:N(t)} - (p-1)X_{N(t)-p+1:N(t)}, \quad p \geq 2$$

is introduced by Thépaut (1950). This treaty is known in actuarial literature as the ECOMOR reinsurance treaty (for further details see e.g. Teugels (1985), Beirlant et al. (1996), Embrechts et al. (1997), Rolski et al. (1999), and Mikosch (2004)).

In general one can define the total loss amount by

$$S_1(p, t) = \sum_{j=1}^p g_j^*(X_{N(t)-j+1:N(t)}),$$

with $g_j^*, 1 \leq j \leq p$ real measurable functions. Typically, the prime interest of a reinsurer is the calculation of the pure premium $\mathbf{E}\{S_1(p, t)\}$. This can be clearly expressed as $\sum_{j=1}^p \mathbf{E}\{g_j^*(X_{N(t)-j+1:N(t)})\}$ supposing additionally that $g_j^*(X_{N(t)-j+1:N(t)}), 1 \leq j \leq p$ have finite expectations. It is often assumed in actuarial applications that $N(t)$ is independent of claim sizes. With that assumption we have for each $j = 1, \dots, p$

$$\mathbf{E}\{g_j^*(X_{N(t)-j+1:N(t)})\} = \mathbf{E}\{\mathbf{E}\{g_j^*(X_{N(t)-j+1:N(t)})|N(t)\}\} = \sum_{k=1}^{\infty} \mathbf{E}\{g_j^*(X_{k-j+1:k})\} \mathbf{P}\{N(t) = k\}.$$

Essentially, in order to compute the pure premium, we need to calculate the expectations of the upper order statistics. Further we need to know the distribution function of the counting random variable $N(t)$. Next, since we consider a bivariate setup, let us suppose further that a similar treaty

$$S_2(q, t) = \sum_{j=1}^q g_j^{**}(Y_{N(t)-j+1:N(t)}), \quad q \geq 1,$$

with $g_j^{**}, 1 \leq j \leq q$ real measurable functions, covers the risks modelled by $Y_1, Y_2, \dots, Y_{N(t)}$.

In order to price both treaties mentioned above, the reinsurer needs to have some indications concerning the distribution function of the total loss amount $(S_1(p, t), S_2(q, t))$. For instance, if in particular the standard deviation (variance) premium principle is used, then an estimate of the standard deviation is required. Further the dependence between $S_1(p, t)$ and $S_2(q, t)$ needs to be quantified.

An asymptotic model for both treaties can be regarded as a good candidate to overcome the difficulties in specification of the model (distribution assumptions for the claim sizes or assumptions on the first and second moment of X_1, Y_1). The idea is to let $t \rightarrow \infty$ and to investigate the joint asymptotic behaviour of $(S_1(p, t), S_2(q, t))$.

Relying on extreme value theory Embrechts et al. (1997) (see Example 8.7.7 therein) derives the asymptotic distribution of both the Largest Claim Reinsurance and the ECOMOR Reinsurance treaties when $N(t)/t \rightarrow$

$\lambda \in (0, \infty)$ in probability. In Hashorva (2004) the asymptotic limiting distribution for $(S_1(p, t), S_2(q, t))$ a bivariate ECOMOR reinsurance treaty is obtained when the marginal distributions of F are in the max-domain of attraction of the Gumbel distribution, imposing further the iid (independent and identically distributed) assumption on the claim sizes.

With main impetus from the afore-mentioned results, we consider in this article special reinsurance treaties with g_j^*, g_j^{**} simple linear functions and claim sizes which can be dependent (thus dropping the iid assumption). In Section 2 we first deal with the joint distribution function of randomly indexed upper order statistics; application to reinsurance and details for the iid case are presented in the Section 3. Proofs of the results are given in Section 4.

2 Joint limiting distribution of randomly indexed upper order statistics

It is well-known (see e.g. Reiss (1989)) that the joint convergence in distribution under the assumption of iid claim sizes

$$\left(\left(\frac{X_{n:n} - b_1(n)}{a_1(n)}, \frac{Y_{n:n} - b_2(n)}{a_2(n)} \right), \dots, \left(\frac{X_{n-m+1:n} - b_1(n)}{a_1(n)}, \frac{Y_{n-m+1:n} - b_2(n)}{a_2(n)} \right) \right) \xrightarrow{d} (\mathcal{X}_1, \mathcal{Y}_1), \dots, (\mathcal{X}_m, \mathcal{Y}_m), \quad n \rightarrow \infty \quad (2.1)$$

with $(\mathcal{X}_i, \mathcal{Y}_i), i \leq m$ random vectors holds for all integers $m \in \mathcal{N}$ and given real functions $a_i(t) > 0, b_i(t), i = 1, 2$ iff the underlying distribution function F of the random vector (X_1, Y_1) is in the max-domain of attraction of H , the max-stable bivariate distribution function of $(\mathcal{X}_1, \mathcal{Y}_1)$, i.e.

$$\lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathcal{R}^2} \left| F^t(a_1(t)x + b_1(t), a_2(t)y + b_2(t)) - H(x, y) \right| = 0 \quad (2.2)$$

holds. See below (2.6) for the joint distribution of $\mathcal{X}_1, \dots, \mathcal{X}_k, k \geq 1$. For short we write the above fact as $F \in MDA(H)$.

We note in passing that the standard notation $\xrightarrow{d}, \xrightarrow{p}, \xrightarrow{a.s.}$ which we use throughout in this paper mean convergence in distribution, convergence in probability and almost sure convergence, respectively.

Actually (2.2) implies that for the marginal distributions of F (denoted here by F_1, F_2) we have $F_i \in MDA(H_i), i = 1, 2$, where the standard extreme value distribution function H_i is either of the following

$$\Phi_{\alpha_i}(x) = \exp(-x^{-\alpha_i}), \quad \alpha_i > 0, \quad x > 0,$$

or

$$\Psi_{\alpha_i}(x) = \exp(-|x|^{\alpha_i}), \quad \alpha_i > 0, \quad x < 0,$$

or

$$\Lambda(x) = \exp(-\exp(-x)), \quad x \in \mathcal{R},$$

i.e. unit Fréchet, Weibull or Gumbel distribution, respectively.

For iid claim sizes, if $N(t) \xrightarrow{P} \infty$ as $t \rightarrow \infty$ and $N(t)$ is independent of the claim sizes for all t large, then by Lemma 2.5.6 of Embrechts et al. (1997) the asymptotic relation (2.2) implies for any $i \geq 1$

$$\left(\frac{X_{N(t)-i+1:N(t)} - b_1(N(t))}{a_1(N(t))}, \frac{Y_{N(t)-i+1:N(t)} - b_2(N(t))}{a_2(N(t))} \right) \xrightarrow{d} (\mathcal{X}_i, \mathcal{Y}_i), \quad t \rightarrow \infty.$$

A different situation arises when transforming with $a_i(t), b_i(t)$ instead of the random functions $a_i(N(t)), b_i(N(t)), i = 1, 2$. So, if we assume further that

$$\frac{N(t)}{t} \xrightarrow{P} \mathcal{Z}, \quad t \rightarrow \infty \quad (2.3)$$

holds with \mathcal{Z} such that $\mathbf{P}\{\mathcal{Z} > 0\} = 1$, then it follows (along the lines of Theorem 4.3.4 of Embrechts et al. (1997)) that for all $i \geq 1$

$$\left(\frac{X_{N(t)-i+1:N(t)} - b_1(t)}{a_1(t)}, \frac{Y_{N(t)-i+1:N(t)} - b_2(t)}{a_2(t)} \right) \xrightarrow{d} (\mathcal{X}_i^*, \mathcal{Y}_i^*), \quad t \rightarrow \infty$$

holds with $(\mathcal{X}_i^*, \mathcal{Y}_i^*)$ a new bivariate random vector.

For the univariate case Theorem 4.3.4 of Embrechts et al. (1997) shows the distribution functions of \mathcal{X}_i^* . Proposition 2.2 of Hashorva (2003) gives an explicit expression for the joint distribution of $(\mathcal{X}_1^*, \dots, \mathcal{X}_i^*)$. Our next result is more general. We consider the bivariate setup allowing claim sizes to be dependent, and further instead of (2.3) we assume the convergence in distribution

$$\frac{N(t)}{t} \xrightarrow{d} \mathcal{Z}, \quad t \rightarrow \infty, \quad (2.4)$$

with \mathcal{Z} almost surely positive. Clearly the above condition is weaker than (2.3).

Proposition 2.1. *Let $\{(X_n, Y_n), n \geq 1\}$ be bivariate claim sizes with common distribution function F independent of the counting process $\{N(t), t \geq 0\}$ for all $t \geq 0$. If condition (2.4) is fulfilled with \mathcal{Z} positive and non-zero and further (2.1) holds for some fixed $m \in \mathcal{N}$ with constants $a_i(t) > 0, b_i(t), i = 1, 2$ that satisfy (2.2), then we have for $t \rightarrow \infty$*

$$\begin{aligned} & \left(\left(\frac{X_{N(t):N(t)} - b_1(t)}{a_1(t)}, \frac{Y_{N(t):N(t)} - b_2(t)}{a_2(t)} \right), \dots, \left(\frac{X_{N(t)-m+1:N(t)} - b_1(t)}{a_1(t)}, \frac{Y_{N(t)-m+1:N(t)} - b_2(t)}{a_2(t)} \right) \right) \\ & \xrightarrow{d} \left((\mathcal{Z}^{\gamma_1} \mathcal{X}_1 + \delta_1, \mathcal{Z}^{\gamma_2} \mathcal{Y}_1 + \delta_2), \dots, (\mathcal{Z}^{\gamma_1} \mathcal{X}_m + \delta_1, \mathcal{Z}^{\gamma_2} \mathcal{Y}_m + \delta_2) \right), \end{aligned} \quad (2.5)$$

with $\gamma_i := 1/\alpha_i, -1/\alpha_i, 0$ if $F_i \in MDA(\Phi_{\alpha_i}), MDA(\Psi_{\alpha_i}), MDA(\Lambda)$, respectively. Further $\delta_i := \ln \mathcal{Z}$ if $F_i \in MDA(\Lambda)$ and 0 otherwise for $i=1, 2$.

In the above proposition we do not assume explicitly the independence of the claim sizes. For iid claim sizes (recall that (2.2) is equivalent to (2.1)) we obtain immediately:

Corollary 2.2. *Let $\{(X_n, Y_n), n \geq 1\}$ be iid bivariate claim sizes with distribution function F independent of $N(t), t \geq 0$. If condition (2.2) is satisfied and further (2.4) holds with \mathcal{Z} almost surely positive, then (2.5) holds for any $m \geq 1$ with $\delta_i, \gamma_i, i = 1, 2$ as in Proposition 2.1.*

Remarks: (i) If $\{X_n, n \geq 1\}$ are iid with distribution function $F \in MDA(H)$, with H a univariate extreme value distribution, then the joint density function of $(\mathcal{X}_1, \dots, \mathcal{X}_m), m \geq 1$ is given by (see e.g. Embrechts et al. (1997) p. 201)

$$h_m(\mathbf{x}) = H(x_m) \prod_{i=1}^m \frac{H'(x_i)}{H(x_i)}, \quad \text{with } x_1 > x_2 > \dots > x_m, \quad \prod_{i=1}^m H(x_i) \in (0, 1). \quad (2.6)$$

Referring to Embrechts et al. (1997) the random vector $(\mathcal{X}_1, \dots, \mathcal{X}_m)$ is called a m -dimensional H -extremal variate. If H is the unit Gumbel distribution, then the following stochastic representation (see Theorem 7.1 of Pakes and Steutel (1997))

$$\left(\mathcal{X}_i - \mathcal{X}_{i+1}\right)_{i=1, \dots, k} \stackrel{d}{=} \left(\frac{E_i}{i}\right)_{i=1, \dots, k}, \quad k \geq 1 \quad (2.7)$$

holds with $E_i, i \geq 1$ iid unit exponential random variables. The standard notation $\stackrel{d}{=}$ means equality of distribution functions. See (3.12) for a stochastic representation of $(\mathcal{X}_1, \dots, \mathcal{X}_m)$.

(ii) In the above proposition \mathcal{Z} is independent of $(\mathcal{X}_i, \mathcal{Y}_i), 1 \leq i \leq m$. This follows immediately recalling that the claim sizes are independent of the counting process $N(t), t \geq 0$.

For the more general class of strictly stationary random sequences conditions for joint weak convergence of upper order statistics are available in literature. For α -mixing stationary sequences (univariate case) the mentioned weak convergence is discussed in Hsing (1988). In Theorem 4.1 and Theorem 4.2 therein necessarily and sufficient conditions for joint weak convergence are derived. Moreover, it is shown that the limit distribution should have a specific form. Novak (2002) gives a simplified version of the limit distribution. A variant of Theorem 4.2 of Hsing (1988) can be found in Leadbetter (1995). Convergence results for dependent bivariate random sequences are obtained in Hüsler (1990, 1993).

3 Joint Asymptotic Distribution of $S_1(p, t)$ and $S_2(q, t)$

In this section we investigate the asymptotic behaviour ($t \rightarrow \infty$) of $(S_1(p, t), S_2(q, t))$ defined in the introduction. In general some restrictions on the choice of the functions $g_j^*, 1 \leq j \leq p, g_j^{**}, 1 \leq j \leq q$ should be imposed. Obviously the total expected loss should be non-negative. Tractable (simple) functions which we consider here are

$$g_j^*(x) = k_{j1}x, \quad 1 \leq j \leq p, \quad \text{and} \quad g_j^{**}(x) = k_{j2}x, \quad 1 \leq j \leq q, \quad (3.8)$$

with $k_{j1}, 1 \leq j \leq p, k_{j2}, 1 \leq j \leq q$ real constants. Put throughout in the following $c_1 := \sum_{j=1}^p k_{j1}, c_2 := \sum_{j=1}^q k_{j2}$. Based on the previous results, we consider now the joint asymptotic behaviour of the reinsurance treaties.

Proposition 3.1. *Let $\{(X_n, Y_n), n \geq 1\}, \{N(t), t \geq 0\}, \mathcal{Z}$ and $\gamma_i, \delta_i, i = 1, 2$ be as in Proposition 2.1, and for $p, q \in \mathcal{N}$ let $g_j^*, 1 \leq j \leq p, g_j^{**}, 1 \leq j \leq q$ be as in (3.8). If condition (2.3) is fulfilled and (2.1) holds for some $m \geq \max(p, q)$, then*

$$\left(\frac{S_1(p, t) - b_1(t)c_1}{a_1(t)}, \frac{S_2(q, t) - b_2(t)c_2}{a_2(t)}\right) \xrightarrow{d} \left(\mathcal{Z}^{\gamma_1} \sum_{j=1}^p k_{j1} \mathcal{X}_j + c_1 \delta_1, \mathcal{Z}^{\gamma_2} \sum_{j=1}^q k_{j2} \mathcal{Y}_j + c_2 \delta_2\right) \quad (3.9)$$

holds as $t \rightarrow \infty$.

Remarks: (i) In the above proposition there is no restriction imposed on the constants k_{ji} . Neither need the claim sizes be positive. Typically, in reinsurance claim sizes are assumed to be positive, and further, some restrictions have to be imposed on the constants k_{ji} . Implicitly we may require that these constants are such that the expected loss (at any time point) for both $S_1(p, t), S_2(q, t)$ is non-negative. A more explicit assumption would be to suppose that both c_1, c_2 are positive.

(ii) Proposition 2.1 and Proposition 3.1 can be shown with similar arguments for a general multivariate setup, i.e. for claim sizes being random vectors in $\mathcal{R}^k, k \geq 3$.

We discuss next the bivariate counterpart of Example 8.7.7 of Embrechts et al. (1997), which was our starting point. We correct a missing constant in an asymptotic result therein.

In order to keep things simple, we impose throughout in the following the assumptions of Corollary 2.2, considering therefore iid claims sizes independent of the counting process $N(t)$.

Note in passing that in Example 8.7.7 of Embrechts et al. (1997) the claim sizes (univariate setup) are iid being further independent of the counting process, which satisfies (2.3) with $\mathcal{Z} = \lambda \in (0, \infty)$ almost surely.

To this end, suppose for simplicity that both marginal distributions F_1, F_2 of F are identical and let F_1^-, F_2^- be the generalised inverse of F_1 and F_2 , respectively.

Case a): If $F_1 \in MDA(\Phi_\alpha), \alpha > 0$, then we may take $b_1(t) = b_2(t) = 0$ and

$$a_1(t) = a_2(t) := \inf\{x \in \mathcal{R} : F_1(x) > 1 - 1/t\} = F_1^-(1 - 1/t), \quad t > 1$$

(see e.g. Resnick (1987), Reiss (1989) or Embrechts et al. (1997)). So we obtain for $p, q \in \mathcal{N}$ and k_{ji} real constants

$$\left(\frac{S_1(p, t)}{a_1(t)}, \frac{S_2(q, t)}{a_2(t)} \right) \xrightarrow{d} \left(\mathcal{Z}^{1/\alpha} \sum_{j=1}^p k_{j1} \mathcal{X}_j, \mathcal{Z}^{1/\alpha} \sum_{j=1}^q k_{j2} \mathcal{Y}_j \right) \quad (3.10)$$

as $t \rightarrow \infty$. Embrechts et al. (1997) proves for the ECOMOR case

$$\frac{S_1(p, n)}{a_1(n)} \xrightarrow{d} \lambda^{1/\alpha} \sum_{i=1}^{p-1} i(\mathcal{X}_i - \mathcal{X}_{i+1}), \quad n \rightarrow \infty$$

which follows immediately from (3.10) putting $\mathcal{Z} = \lambda > 0$.

Case b): When $F_1 \in MDA(\Psi_\alpha), \alpha > 0$ holds, then we may take $b_1(t) = b_2(t) := \omega, t > 1$, with $\omega := \sup\{x : F_1(x) < 1\}$ the upper endpoint of the distribution function F_1 which is necessarily finite and define

$$a_1(t) = a_2(t) := \omega - F_1^-(1 - 1/t), \quad t > 1.$$

Thus we have

$$\left(\frac{S_1(p, t) - c_1 \omega}{a_1(t)}, \frac{S_2(q, t) - c_2 \omega}{a_2(t)} \right) \xrightarrow{d} \left(\mathcal{Z}^{-1/\alpha} \sum_{j=1}^p k_{j1} \mathcal{X}_j, \mathcal{Z}^{-1/\alpha} \sum_{j=1}^q k_{j2} \mathcal{Y}_j \right), \quad t \rightarrow \infty.$$

Case c): If $F_1 \in MDA(\Lambda)$, then we put for t large

$$b_1(t) = b_2(t) := F_1^-(1 - 1/t)$$

and

$$a_1(t) = a_2(t) := \int_{b_1(t)}^{\omega} [1 - F_1(s)]/[1 - F_1(b_1(t))] ds.$$

Thus we have that the right hand side of (3.9) is given by

$$\left(\sum_{j=1}^p k_{j1} \mathcal{X}_j + c_1 \ln \mathcal{Z}, \sum_{j=1}^q k_{j2} \mathcal{Y}_j + c_2 \ln \mathcal{Z} \right).$$

Next, we consider 3 examples.

Example 1. (Generalised ECOMOR reinsurance treaty). Assume that the constants k_{ij} are such that $c_1 = c_2 = 0$. This is fulfilled in the special case of the ECOMOR treaty (see introduction above).

Thus under the assumptions of Proposition 3.1 we have for $F_1, F_2 \in MDA(\Lambda)$ the convergence in distribution

$$\left(\frac{S_1(p, t)}{a_1(t)}, \frac{S_2(q, t)}{a_2(t)} \right) \xrightarrow{d} \left(\sum_{j=1}^p k_{j1} \mathcal{X}_j, \sum_{j=1}^q k_{j2} \mathcal{Y}_j \right), \quad t \rightarrow \infty.$$

For the ECOMOR treaty Embrechts et al. (1997) obtains with $\mathcal{Z} = \lambda$ almost surely

$$\frac{S_1(p, n)}{a_1(n)} \xrightarrow{d} \sum_{j=1}^{p-1} j(\mathcal{X}_j - \mathcal{X}_{j+1}) \stackrel{d}{=} \sum_{j=1}^{p-1} E_j, \quad n \rightarrow \infty,$$

with $E_j, 1 \leq j \leq p-1$ iid unit exponential random variables, which follows also by the bivariate result above (recall (2.7)).

There is a remarkable fact in the above asymptotic result, namely the random variable \mathcal{Z} does not appear in the right hand side of the asymptotic expression. This is not the case in general for $F_i, i = 1, 2$ in the max-domain of attraction of Fréchet or Weibull.

Example 2. (Asymptotic independence of the components of the maximum claim sizes.) As mentioned in the introduction, for the asymptotic considerations, we are not directly interested in the joint distribution function F of the claim sizes, but on the limiting distribution H . In some applications, even if F is not a product distribution, it may happen that H is a product distribution, meaning $H(x, y) = H_1(x)H_2(y), \forall x, y \in \mathcal{R}$. This implies that \mathcal{X}_i is independent of \mathcal{Y}_j for any $i, j \geq 1$ and thus $[S_1(p, t) - b_1(t)c_1]/a_1(t)$ is asymptotically independent of $[S_2(q, t) - b_2(t)c_2]/a_2(t)$ as $t \rightarrow \infty$. So the asymptotic distribution of each treaty can be easily calculated using (2.6). We note in passing that there are several known conditions for asymptotic independence, see e.g. Galambos (1987), Resnick (1987), Reiss (1989), Falk et al. (1994), Hüsler (1994).

Example 3. (F_1, F_2 with exponential tails and $N(t)$ Poisson). Consider iid claim sizes with joint distribution function F which has marginal distributions tail equivalent to the unit exponential distribution function, i.e. $\lim_{x \rightarrow \infty} \exp(x)[1 - F_i(x)] = 1, i = 1, 2$. It follows that $F_1, F_2 \in MDA(\Lambda)$ with constants $a_1(t) = a_2(t) = 1, b_1(t) = b_2(t) = \ln t, t > 0$. As in the above example, assume further that the distribution function H is a product distribution and $N(t)$ is a (homogeneous) Poisson process with parameter $\lambda > 0$ independent of the claim sizes. We have thus $N(t)/t \xrightarrow{a.s.} \lambda$ and for any $p, q \in \mathcal{N}$

$$\left(S_1(p, t) - c_1 \ln t, S_2(q, t) - c_2 \ln t \right) \xrightarrow{d} \left(\sum_{j=1}^p k_{j1} \mathcal{X}_j + c_1 \ln \lambda, \sum_{j=1}^q k_{j2} \mathcal{Y}_j + c_2 \ln \lambda \right), \quad t \rightarrow \infty,$$

with \mathcal{X}_j independent of \mathcal{Y}_j . Recall $c_1 := \sum_{j=1}^p k_{j1}, c_2 := \sum_{j=1}^q k_{j2}$.

Borrowing the idea of Example 8.7.7 of Embrechts et al. (1997) we find an explicit formula for the right hand side above. Let therefore $E_{ji}, j \geq 1, i = 1, 2$ be iid unit exponential random variables and put $\bar{k}_{li} := \sum_{j=1}^l k_{ji}/l, l \geq 1, i = 1, 2$. It is well-known that (see e.g. Reiss (1989))

$$\left(E_{n-j+1:n,i} \right)_{j=1,\dots,n} \stackrel{d}{=} \left(\sum_{l=j}^n \frac{E_{li}}{l} \right)_{j=1,\dots,n}, \quad n \geq 1, i = 1, 2.$$

It is well-known that for large n

$$\sum_{l=1}^n \frac{1}{l} - \ln n = K + o(1)$$

where K is the Euler-Mascheroni constant. So we may write for $n \geq p \geq 1$

$$\begin{aligned}
\sum_{j=1}^p k_{j1} (E_{n-j+1:n,1} - \ln n) &\stackrel{d}{=} \sum_{j=1}^p k_{j1} \sum_{l=j}^n \frac{E_{l1}}{l} - c_1 \ln n \\
&= \sum_{l=1}^p E_{l1} \frac{1}{l} \left(\sum_{j=1}^l k_{j1} \right) + c_1 \sum_{l=p+1}^n \frac{E_{l1}}{l} + c_1 \left(K - \sum_{l=1}^n \frac{1}{l} \right) + o(1) \\
&\stackrel{d}{\rightarrow} \sum_{l=1}^p \bar{k}_{l1} E_{l1} + c_1 \sum_{l=p+1}^{\infty} \frac{E_{l1} - 1}{l} + c_1 \left(K - \sum_{l=1}^p \frac{1}{l} \right), \quad n \rightarrow \infty.
\end{aligned}$$

On the other hand

$$(E_{n:n,1} - \ln n, \dots, E_{n-j+1:n,1} - \ln n) \stackrel{d}{\rightarrow} (\mathcal{X}_1, \dots, \mathcal{X}_j), \quad n \rightarrow \infty,$$

hence the continuous mapping theorem (see e.g. Kallenberg (1997)) implies

$$\sum_{j=1}^p k_{j1} (E_{n-j+1:n,1} - \ln n) \stackrel{d}{\rightarrow} \sum_{j=1}^p k_{j1} \mathcal{X}_j, \quad n \rightarrow \infty.$$

Thus we have the stochastic representation

$$\sum_{j=1}^p k_{j1} \mathcal{X}_j \stackrel{d}{=} \sum_{l=1}^p \bar{k}_{l1} E_{l1} + c_1 \sum_{l=p+1}^{\infty} \frac{E_{l1} - 1}{l} + c_1 K_p, \quad (3.11)$$

with $K_i := K - \sum_{l=1}^i \frac{1}{l}$, $i \geq 1$. We obtain thus proceeding similarly for the second treaty and recalling the asymptotic independence assumption

$$\begin{aligned}
&\left(S_1(p, t) - c_1 \ln t, S_2(q, t) - c_2 \ln t \right) \\
&\stackrel{d}{\rightarrow} \left(\sum_{j=1}^p \bar{k}_{j1} E_{j1} + c_1 \left[\sum_{j=p+1}^{\infty} \frac{E_{j1} - 1}{j} + K_p + \ln \lambda \right], \sum_{j=1}^q \bar{k}_{j2} E_{j2} + c_2 \left[\sum_{j=q+1}^{\infty} \frac{E_{j2} - 1}{j} + K_q + \ln \lambda \right] \right).
\end{aligned}$$

Remarks. (i) For a suitable choice of constants (3.11) implies for any $m \geq 2$

$$(\mathcal{X}_1, \dots, \mathcal{X}_m) \stackrel{d}{=} \left(E_{11} + \sum_{l=2}^{\infty} \frac{E_{l1} - 1}{l} + K_1, \dots, E_{m1} + \sum_{l=m+1}^{\infty} \frac{E_{l1} - 1}{l} + K_m \right), \quad (3.12)$$

hence in particular we have for any $i \in \mathcal{N}$

$$\mathbf{E}\{\mathcal{X}_i\} = 1 + K_i, \quad \mathbf{Var}\{\mathcal{X}_i\} = 1 + \sum_{l=i+1}^{\infty} \frac{1}{l^2} = \frac{\pi^2}{6} + 1 - \sum_{l=1}^i \frac{1}{l^2} < \infty.$$

(ii) In Example 8.7.7 of Embrechts et al. (1997) the term $-k \ln k$ in page 519 should be $k(K - \sum_{j=1}^k \frac{1}{j})$.

A general result is given in the following proposition.

Proposition 3.2. *Let $\{(X_n, Y_n), n \geq 1\}$ be iid bivariate claim sizes with distribution function F . Assume that (2.2) holds with given functions $a_i(t) > 0, b_i(t), i = 1, 2$ and $H(x, y) = \exp(-\exp(-x) - \exp(-y)), x, y \in \mathcal{R}$. Suppose further that $\{N(t), t \geq 0\}$ is a counting process independent of the claim sizes satisfying (2.4) with \mathcal{Z} positive and non-zero. Then we have the convergence in distribution*

$$\left(\frac{S_1(p, t) - b_1(t)c_1}{a_1(t)}, \frac{S_2(q, t) - b_2(t)c_2}{a_2(t)} \right)$$

$$\xrightarrow{d} \left(\sum_{j=1}^p \bar{k}_{j1} E_{j1} + c_1 \left[\sum_{j=p+1}^{\infty} \frac{E_{j1}-1}{j} + \ln \mathcal{Z} + K_p \right], \sum_{j=1}^q \bar{k}_{j2} E_{j2} + c_2 \left[\sum_{j=q+1}^{\infty} \frac{E_{j2}-1}{j} + \ln \mathcal{Z} + K_q \right] \right) \quad (3.13)$$

as $t \rightarrow \infty$, with $c_i, \bar{k}_{ji}, K_p, K_q$ as defined above and $E_{ji}, j \geq 1, i = 1, 2$ iid unit exponential random variables.

4 Proofs

Proof of Proposition 2.1: Define in the following for t positive

$$\mathbf{T}_j(N(t)) := (T_{j1}(N(t)), T_{j2}(N(t))) = \left(\frac{X_{N(t)-j+1:N(t)} - b_1(N(t))}{a_1(N(t))}, \frac{Y_{N(t)-j+1:N(t)} - b_2(N(t))}{a_2(N(t))} \right)$$

and

$$Z_t := \frac{N(t)}{t}, \quad \tilde{a}_i(t, z) := \frac{a_i(tz)}{a_i(t)}, \quad \tilde{b}_i(t, z) := \frac{b_i(tz) - b_i(t)}{a_i(t)}, \quad z > 0, i = 1, 2.$$

Let $z_t, t \geq 0$ be arbitrary positive constants such that $\lim_{t \rightarrow \infty} z_t = z \in (0, \infty)$. Assumption (2.2) implies that the positive norming functions a_1, a_2 are regularly varying (see e.g. Resnick (1987)), hence using Theorem A3.2 of Embrechts et al. (1997) we obtain

$$\lim_{t \rightarrow \infty} \tilde{a}_i(t, z_t) = z^{\gamma_i}, \quad i = 1, 2,$$

with $\gamma_i := 1/\alpha_i, -1/\alpha_i, 0$ if $F_i \in MDA(\Phi_{\alpha_i}), F_i \in MDA(\Psi_{\alpha_i})$ or $F_i \in MDA(\Lambda)$, respectively.

If $F \in MDA(\Psi_{\alpha})$ or $F \in MDA(\Phi_{\alpha})$ with $\alpha > 0$, then for large t we have $b_i(ts) - b_i(t) = 0, i = 1, 2, \forall s > 0$. It is well-known (see Proposition 0.10, Proposition 1.1, Corollary 1.7 and Exercise 0.4.3.2 of Resnick (1987)) that if $F \in MDA(\Lambda)$ then $\lim_{t \rightarrow \infty} (b_i(ts) - b_i(t))/a_i(t) = \ln s, s > 0$ holds locally uniformly on $(0, \infty)$. Consequently we have

$$\lim_{t \rightarrow \infty} \tilde{b}_i(t, z_t) = \delta_i \ln z, \quad i = 1, 2,$$

with $\delta_i = 1$ if $F_i \in MDA(\Lambda)$ and $\delta_i = 0$, otherwise.

In view of (2.4) applying now Theorem 3.27 of Kallenberg (1997) we obtain the convergence in distributions

$$\tilde{a}_i(t, Z_t) \xrightarrow{d} \mathcal{Z}^{\gamma_i}, \quad \text{and} \quad \tilde{b}_i(t, Z_t) \xrightarrow{d} \delta_i \ln \mathcal{Z}, \quad t \rightarrow \infty, i = 1, 2,$$

hence as $t \rightarrow \infty$

$$\mathbf{U}_{t, N(t)} := (\tilde{a}_1(t, Z_t), \tilde{a}_2(t, Z_t), \tilde{b}_1(t, Z_t), \tilde{b}_2(t, Z_t)) \xrightarrow{d} (\mathcal{Z}^{\gamma_1}, \mathcal{Z}^{\gamma_2}, \delta_1 \ln \mathcal{Z}, \delta_2 \ln \mathcal{Z}) =: \mathbf{U}_{\mathcal{Z}}.$$

Let $\mathbf{w}_j \in \mathcal{R}^2, 1 \leq j \leq m$ and $\mathbf{u} \in (0, \infty)^2 \times \mathcal{R}^2$ be given constants. Since the claim sizes are independent of $N(t), t \geq 0$ we obtain by conditioning

$$\mathbf{P}\{\mathbf{U}_{t, N(t)} \leq \mathbf{u}, \mathbf{T}_1(N(t)) \leq \mathbf{w}_1, \dots, \mathbf{T}_m(N(t)) \leq \mathbf{w}_m | N(t) = n\} = \mathbf{1}(\mathbf{U}_{t, n} \leq \mathbf{u})h(n), \quad n \geq m, t > 0,$$

with $h(n) := \mathbf{P}\{\mathbf{T}_1(n) \leq \mathbf{w}_1, \dots, \mathbf{T}_m(n) \leq \mathbf{w}_m\}$. Here $\mathbf{a} \leq \mathbf{b}, \mathbf{a}, \mathbf{b} \in \mathcal{R}^k, k \geq 2$ is understood component-wise and $\mathbf{1}(\cdot)$ is the indicator function. By the assumptions

$$\lim_{n \rightarrow \infty} h(n) = \mathbf{P}\{(\mathcal{X}_1, \mathcal{Y}_1) \leq \mathbf{w}_1, \dots, (\mathcal{X}_m, \mathcal{Y}_m) \leq \mathbf{w}_m\}.$$

Using (2.4) we obtain $N(t) \xrightarrow{P} \infty$ as $t \rightarrow \infty$ implying

$$h(N(t)) \xrightarrow{P} \mathbf{P}\{(\mathcal{X}_1, \mathcal{Y}_1) \leq \mathbf{w}_1, \dots, (\mathcal{X}_m, \mathcal{Y}_m) \leq \mathbf{w}_m\}, \quad t \rightarrow \infty.$$

If further \mathbf{u} is such that $\mathbf{P}\{U_{\mathcal{Z}} = \mathbf{u}\} = 0$ then

$$\mathbf{1}(U_{t,N(t)} \leq \mathbf{u}) \xrightarrow{d} \mathbf{1}(U_{\mathcal{Z}} \leq \mathbf{u}), \quad t \rightarrow \infty,$$

consequently by Slutsky lemma (see e.g. Kallenberg (1997))

$$\mathbf{1}(U_{t,N(t)})h(N(t)) \xrightarrow{d} \mathbf{1}(U_{\mathcal{Z}} \leq \mathbf{u})\mathbf{P}\{(\mathcal{X}_1, \mathcal{Y}_1) \leq \mathbf{w}_1, \dots, (\mathcal{X}_m, \mathcal{Y}_m) \leq \mathbf{w}_m\}, \quad t \rightarrow \infty.$$

Since $\mathbf{1}(U_{t,N(t)})h(N(t)), t > 0$ is positive and bounded (consequently uniformly integrable) we have taking the expectation with respect to $N(t)$ and passing to limit

$$\left(U_{t,N(t)}, T_1(N(t)), \dots, T_m(N(t)) \right) \xrightarrow{d} \left(U_{\mathcal{Z}}, (\mathcal{X}_1, \mathcal{Y}_1), \dots, (\mathcal{X}_m, \mathcal{Y}_m) \right), \quad t \rightarrow \infty.$$

Now, for any $j \leq m, t > 0$ we may write

$$\begin{aligned} & \left(\left(\frac{X_{N(t)-j+1:N(t)} - b_1(t)}{a_1(t)}, \frac{Y_{N(t)-j+1:N(t)} - b_2(t)}{a_2(t)} \right) \right) \\ &= \left(\tilde{a}_1(t, Z_t)T_{j1}(N(t)) + \tilde{b}_1(t, Z_t), \tilde{a}_2(t, Z_t)T_{j2}(N(t)) + \tilde{b}_2(t, Z_t) \right), \end{aligned}$$

hence the proof follows by the continuous mapping theorem. \square

Proof of Proposition 3.1: By the assumptions and Proposition 2.1 we have that (2.5) holds, hence we may write for $t > 0, p, q \in \mathcal{N}$ using the continuous mapping theorem

$$\begin{aligned} & \left(\frac{S_1(p, t) - b_1(t)c_1}{a_1(t)}, \frac{S_2(q, t) - b_2(t)c_2}{a_2(t)} \right) \\ &= \left(\sum_{j=1}^p k_{j1} [X_{N(t)-j+1:N(t)} - b_1(t)]/a_1(t), \sum_{j=1}^q k_{j2} [Y_{N(t)-j+1:N(t)} - b_2(t)]/a_2(t) \right) \\ &\xrightarrow{d} \left(\mathcal{Z}^{\gamma_1} \sum_{j=1}^p k_{j1} \mathcal{X}_j + c_1 \delta_1, \mathcal{Z}^{\gamma_2} \sum_{j=1}^q k_{j2} \mathcal{Y}_j + c_2 \delta_2 \right), \quad t \rightarrow \infty, \end{aligned}$$

with $\delta_i, \gamma_i, i = 1, 2$ as in Proposition 2.1, thus the proof is complete. \square

Proof of Proposition 3.2: The proof follows immediately using Proposition 3.1 and the result of Example 3 recalling further the expression for the joint density function given in (2.6). \square

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